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UTILITY, ENTROPY AND A "PARADOX" OF TRAFFIC FLOW

by

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ABSTRACT

The relation between various derivations of the logit based route choice model and the associated benefit measures are explored in the light of "another" paradox of traffic flow highlighted recently by Sheffi and Daganzo.

1. INTRODUCTION

Considerable attention has been paid in the transportation research literature to the interpretation of counter-intuitive flows and travel times that result from the application of particular traffic assignment models after the addition of a link to the network or the modification of its length. Of note is the so called Braess paradox (Braess, 1968; Murchland, 1970) which occurs for networks with flow dependant link costs (travel times); the addition of a link may result in a "user-optimized" flow distribution over the routes of the network which increases the total travel costs. This phenomenon is well understood, since finding the user-optimized flows is equivalent to solving a minimization problem with an objective function which does not equal to the total travel costs.

Also, it has been observed that the application of multipath stochastic assignment methods to network with fixed link costs (Von Falkenhausen, 1966; Burnell, 1968; Dial, 1971; Gunnarson, 1972) leads to counter-intuitive flows and average travel costs. The use of the logistic function in such methods results in the assignment of unreasonably high flows to overlapping routes (Schneider, 1973; Burrell, 1976; Florian and Fox, 1976). This defficiency may be remedied by the use of probit path choice as pointed out by Daganzo and Sheffi (1978), although large scale problems may not yet be handled by the latter method.

More recently, Sheffi and Daganzo (1978) pointed out that the reduction of the cost of a link or the addition of a link may result in flows (obtained from the application of Dial's method) that increase the overall travel time, a phenomenon similar to Braess' paradox. Williams (1977) noted that

this phenomenon may occur with stochastic choice models that are not based on the logit function. (See also Harris and Tanner, 1974; Lerman, 1975; Ben Akiva and Lerman, 1977). Williams proposed the use of the expected *perceived* minimum travel cost, a quantity which may be shown to be monotonic with respect to the cost of each path. This is also advocated by Sheffi and Daganzo who use this quantity to resolve the apparent paradox of the stochastic assignment method.

The purpose of this note is to offer a different utility based derivation of the stochastic assignment method of Dial and to show that the expected perceived minimum travel time, that is derived from random utility theory arguments, may be shown to be equal to a dual variable of a constrained maximization problem with an objective function which is the average utility of travellers using the available routes of the network. Since this objective function is intimately related to entropy maximization, some general conclusions are drawn on the relation between the properties of models based on utility maximization principles and on entropy maximization. Some of these conclusions are implicit in the work of Williams (1977).

The paradox of stochastic route choice, as presented by Sheffi and Daganzo is briefly revisited in the next section.

2. THE "ANOTHER" PARADOX OF TRAFFIC FLOW

Consider a network consisting of two distinct and disjoint routes of length c_1 and c_2. The logit flow allocations in Dial's algorithm assigns to route 1 the fraction $\frac{\exp(-\theta c_1)}{\sum_{j=1}^{2} \exp(-\theta c_j)}$ of total trips and to route 2 the remainder of the

trips. The average travel time (or expected cost per trip ($\bar{c}$ is given by the expression

$$\bar{c} = \frac{c_2 + c_1 \exp(\theta(c_2 - c_1))}{1 + \exp(\theta(c_2 - c_1))} \tag{1}$$

which is not a monotone function of c_2 or c_1. $\bar{c}$ achieves its maximum at $(c_2 - c_1) = \frac{1.28}{\theta}$ and decreases if c_2 is increased for fixed c_1 if

$$c_2 - c_1 > \frac{1.28}{\theta} \tag{2}$$

Thus, for a given c_1, if c_2 is decreased (link improvement) and both the initial and improved values satisfy (2), then $\bar{c}$ increases, a most counter-intuitive property. This peculiarity is due strictly to the logit flow allocation rule.

The second paradoxal property pointed out by Sheffi and Daganzo is related to link or route additions. Suppose that a third route is added and the cost of this route is c_3, $c_3 > \max(c_1, c_2)$. (This route may be distinct from the other routes or may share some links). Then, the average time $\bar{c}$ increases as well, since the probability of using this longer route is non zero! This peculiarity is not necessarily due to the logit flow allocation rule, but would occur with *any* traffic assignment procedure based on stochastic flow allocation derived from random utility theory principles: the probability that the longer route 3 is perceived as the shortest route is non zero, due to the stochastic distribution of the perception errors.

The greatest inconvenience posed by these properties occurs where the total travel time is used to evaluate the viability of network expansion; the seemingly logical conclusion would be to do nothing whenever $\bar{c}$

increases. Thus consistency with random utility theory imposes the use of the expected perceived minimum travel time, $E(\tilde{t})$, where

$$\tilde{t} = \min_j \{t_j\} \tag{3}$$

and t_j is the perceived cost of route j of measured cost c_j. The main result is that

$$E\{\tilde{t}\} = -\frac{1}{\theta} \ell n \sum_j \exp(-\theta c_j) \tag{4}$$

Williams provides the most general derivation of this result. The satisfying property of (4) is that it is monotonically increasing with respect to the measured travel cost c_j, thus its use would do away with the first paradoxical behavior of the logit flow allocation rule; the less satisfying property of (4) is that it is monotonically decreasing with the size of the choice set. Thus it may only be used to compare two distinct link additions and cannot be used to compare a link addition with the "do-nothing" alternative. Be that as it may, the use of (4) as an evaluation measure is consistent with the random utility derivation of the stochastic traffic assignment methods.

In the following section we show that the logit based stochastic assignment method and (4) may be derived by using a deterministic utility maximization approach.

3. ANOTHER DERIVATION OF THE LOGIT BASED STOCHASTIC ASSIGNMENT

Consider the following interpretation of the route choice problem as perceived by an individual who takes *daily* trips from an origin (zone) to a destination (zone), such as the work trips. Let us suppose that over a period of time he will take T trips, for which he may select among n routes

and that this selection is governed by a rational behavior which implies the maximization of his net utility $U(T_1,\ldots,T_2,\ldots,T_n) - \sum_k c_k T_k$, where T_k is the number of trips that he will take on route k. Clearly, $\sum_k T_k = T$. If the utility function is linear or convex, he will *always* select one route, since the net utility will be maximized by an extremal choice. On the other hand, if this utility function is concave, then he may select, over the said period of time, more than one route. In particular, if his utility function is the integrated log function, $U = U_0 + \sum_k [\beta . T_k - \int_1^{T_k} \ell n\ t\ dt]$, introduced by Beckmann and Golob (1971) for a utility based derivation of the gravity trip distribution problem, the vector of trips $\{T_k\}$ taken by an individual is the solution to the problem

$$\text{Max}\quad U_0 + \sum_k [(\beta + \beta_0)\ T_k - \beta_0\ (T_k\ \ell n\ T_k + 1)] - \sum_k c_k . T_k \qquad (5)$$

$$\text{subject to}\quad \sum_k T_k = T \qquad (6)$$

$$T_k \geq 0\ ,\quad k=1,\ldots,n \qquad (7)$$

It is easy to show, by using the Lagrangean approach or Kuhn-Tucker theory that

$$T_k = \exp\ \frac{-c_k}{\beta_0} + \frac{(\beta - u)}{\beta_0} > 0 \qquad (8)$$

where u is the dual variable associated with equation (5). Thus, the frequency, or proportion, of trips taken on route k is

$$f_k = \frac{T_k}{\sum_j T_j} = \frac{\exp\ (-\theta c_k)}{\sum_j \exp\ (-\theta c_j)} \qquad (9)$$

where θ is a constant equal to $\frac{1}{\beta_0}$. If we suppose that the route choice of an individual may be observed, then θ is in principle a parameter which may be calibrated from data.

Now, the rest of the argumentation is transparent and may be anticipated by an informed reader. Let all the g travellers between the origin and the destination be governed by the same utility function. Thus, the resulting *daily* average flows h_k the available routes k are given by

$$h_k = g \cdot \frac{\exp(-\theta c_k)}{\sum_j \exp(-\theta c_j)} \quad , \quad \text{all routes } k \tag{10}$$

and the link flows v_a will be given by

$$v_a = \sum_a \delta_{ak} \cdot h_k \tag{11}$$

where δ_{ak} is an indicator function which equals 1 if arc a belongs to route k and nil otherwise. Now, it is possible to find (see also Fisk 1978) an equivalent maximization problem which is equivalent to solving (10) and (11). In particular

$$\text{Max} \quad H_0 + \sum_j [(\beta + \beta_0) h_j - \beta_0 (h_j \ln h_j + 1)] - c_j h_j \tag{12}$$

$$\text{subject to} \quad \sum_j h_j = g \tag{13}$$

$$h_j \geq 0 \tag{14}$$

and (10).

A plausible interpretation of the objective function (12) is that due to the derivation of (10) it is equivalent to the maximization of the net average aggregate (daily) utility of travellers g and thus, the dual variable associated with constraint (13) is a proper measure of the change in benefit to these travellers when, the set of available routes, the route costs or the total demand change.

It is relatively easy to show that

$$u = \frac{1}{\theta} \ell n\ h_k + c_k + \frac{1}{\theta} \tag{15}$$

and thus, by substituting (10) for h_k we obtain

$$u = -\frac{1}{\theta} \ell n \sum_j \exp(-\theta c_j) + (\ell n\ g - 1)/\theta \tag{16}$$

Since the last term of (16) is constant if the total demand g remains constant, it may be disregarded when measuring changes in benefit due changes of routes and their costs. Thus, we have found the same measure of benefit as (4) by using a deterministic utility maximization approach.

Note that the same result may be obtained by replacing (11) with

$$\text{Max} - \sum_j \frac{1}{\theta} h_j \ell n\ h_j - \sum_j c_j \cdot h_j \tag{17}$$

which is the maximization of the entropy of the route flows. This is rather well known by now, however it is relevant to emphasize that the derivation of (17) does not make any assumptions regarding the source of variability of route choice and does not offer any causal interpretation.

The implications of the multiple ways in which the logit based route choice model may be derived are explored briefly in the next section.

4. DISCUSSION

The multiple ways in which the logistics route choice model and its proper evaluation measures may be derived, merely underline the fact that the functional choices made in various theoretical approaches often dwarf the

respectability of the theoretical underpinnings of the method. Thus, one may paradoxically conclude that the assumption that the error terms associated with the perception of the route costs, by *all travelers*, have an extreme type 1 distribution with zero mean and variance one, in the random utility approach is somehow equivalent to supposing that *all travellers* are governed by the same integrated log utility function in the deterministic utility approach; furthermore both approaches are somehow equivalent to using the entropy maximization approach, which makes no assumption whatsoever about the sources of variability and generates in principle, a descriptive model. However in the case when one may attribute the observed behavior to several sources of variability, the observed route choice phenomena surpass the theoretical properties of any of the above derivations.

For instance, when Sheffi and Daganzo interpret (4) by supposing that "if one interviewed the users of the system one would find that the total *perceived* cost per user would have decreased!" and that "users of the system will base their choice on minimization of perceived cost, the basic principle of random utility models", they trust that all travellers behave in fact as random utility theory supposes them to do. Since the benefit measure may be derived from other utility maximization arguments, it is clear that other interpretations are possible.

On the other hand, whatever the derivation, the counter-intuitive properties of these models prevail, and their use is often justified by computational facility, an often misused attribute of a transportation planning model.

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